1. 给定一个随机过程 $X(t)$ 和常数 a，试用 $X(t)$ 的相关函数表示随机过程

$$Y(t)=X(t+a)-X(t)$$

的相关函数.

解：$R_Y(t_1,t_2)=E[Y(t_1)Y(t_2)]$

$=E[X(t_1+a)-X(t_1)][X(t_2+a)-X(t_2)]$

$=EX(t_1+a)X(t_2+a)-EX(t_1+a)X(t_2)-EX(t_1)X(t_2+a)+EX(t_1)X(t_2)$

$=R_X(t_1+a,t_2+a)-R_X(t_1+a,t_2)-R_X(t_1,t_2+a)+R_X(t_1,t_2)$

2. 设随机过程

$$X(t)=A\cos(\omega_0 t+\Phi),\ -\infty<t<+\infty$$

其中，ω_0 为正常数，A 和 Φ 是相互独立的随机变量，且 A 服从在区间$[0,1]$上的均匀分布，而 Φ 服从在区间$[0,2\pi]$上的均匀分布. 试求 $X(t)$ 的数学期望和相关函数.

解：$f(a)=\begin{cases}1 & 0<a<1\\ 0 & 其它\end{cases}$　　$f(\varphi)=\begin{cases}\frac{1}{2\pi} & 0<\varphi<2\pi\\ 0 & 其它.\end{cases}$

$m_X(t)=EX(t)=E[A\cos(w_0t+\Phi)]$

$=\int_0^1\int_0^{2\pi}a\cos(w_0t+\varphi)\cdot 1\cdot\frac{1}{2\pi}\,da\,d\varphi$

$=\frac{1}{2\pi}\int_0^1 a\,da\int_0^{2\pi}\cos(w_0t+\varphi)\,d\varphi$

$=\frac{1}{2\pi}\cdot\frac{1}{2}a^2\Big|_0^1\cdot\sin(w_0t+\varphi)\Big|_0^{2\pi}$

$=\frac{1}{2\pi}\cdot\frac{1}{2}\cdot(\sin(w_0t+2\pi)-\sin(w_0t))]$

$=0$

$R_X(t_1,t_2)=EX(t_1)X(t_2)=E[A\cos(w_0t_1+\Phi)\cdot A\cos(w_0t_2+\Phi)]$

$=\int_0^1\int_0^{2\pi}a^2\cos(w_0t_1+\varphi)\cos(w_0t_2+\varphi)\cdot 1\cdot\frac{1}{2\pi}\,da\,d\varphi$

$=\frac{1}{2\pi}\int_0^1 a^2da\int_0^{2\pi}\cos(w_0t_1+\varphi)\cos(w_0t_2+\varphi)\,d\varphi$

$=\frac{1}{2\pi}\cdot\frac{1}{3}\cdot\int_0^{2\pi}\frac{\cos(w_0(t_1+t_2)+2\varphi)+\cos w_0(t_1-t_2)}{2}\,d\varphi$

$=\frac{1}{12\pi}\left(\frac{1}{2}\sin[w_0(t_1+t_2)+2\varphi]\Big|_0^{2\pi}+\cos w_0(t_1-t_2)\varphi\Big|_0^{2\pi}\right)$

$=\frac{1}{12\pi}(0+2\pi\cos w_0(t_1-t_2))$

$=\frac{1}{6}\cos w_0(t_1-t_2)$

3. 设随机过程 $X(t)=X+Yt+Zt^2$，$-\infty<t<+\infty$，其中，X,Y,Z 是相互独立的随机变量，各自的数学期望为零，方差为 1. 试求 $X(t)$ 的协方差函数.

解：$EX(t)=EX+tEY+t^2EZ$

$=0$

$C_X(t_1,t_2)=EX(t_1)X(t_2)-EX(t_1)EX(t_2)$

$=EX(t_1)X(t_2)-0$

$=E[X+Yt_1+Zt_1^2][X+Yt_2+Zt_2^2]$

$=E[X^2+XYt_2+XZt_2^2+XYt_1+Y^2t_1t_2+YZt_1t_2^2+XZt_1^2+YZt_1^2t_2+Z^2t_1^2t_2^2]$

$=0.$

$=EX^2+t_1t_2EY^2+t_1^2t_2^2EZ^2$

$=1+t_1t_2+t_1^2+t_2^2$

4. 设随机过程 $X(t)$ 的导数存在，试证

$$E\left[X(t)\frac{dX(t)}{dt}\right]=\left.\frac{\partial R_X(t_1,t)}{\partial t_1}\right|_{t_1=t}$$

证明：$\because \dfrac{dX(t)}{dt}=\lim\limits_{h\to0}\dfrac{X(t+h)-X(t)}{h}$

$\therefore X(t)\dfrac{dX(t)}{d(t)}=X(t)\lim\limits_{h\to0}\dfrac{X(t+h)-X(t)}{h}=\lim\limits_{h\to0}\dfrac{X(t)X(t+h)-X^2(t)}{h}$

$\therefore E\left[X(t)\dfrac{dX(t)}{dt}\right]=E\left[\lim\limits_{h\to0}\dfrac{X(t)X(t+h)-X^2(t)}{h}\right]=\lim\limits_{h\to0}\dfrac{R_X(t,t+h)-R_X(t,t)}{h}$

$=\left.\dfrac{\partial R_X(t_1,t)}{\partial t_1}\right|_{t_1=t}$

5. 试证均方导数的下列性质：

(1) $E\left[\dfrac{dX(t)}{dt}\right]=\dfrac{dEX(T)}{dt}$；

证明：$\because \dfrac{dX(t)}{dt}=\lim\limits_{h\to0}\dfrac{X(t+h)-X(t)}{h}$

$\therefore E\left[\dfrac{dX(t)}{dt}\right]=E\left[\lim\limits_{h\to0}\dfrac{X(t+h)-X(t)}{h}\right]$

$=\lim\limits_{h\to0}\dfrac{EX(t+h)-EX(t)}{h}$

$=\dfrac{dEX(t)}{dt}$

6. 试证均方积分的下列性质：

(1) $E\left[\int_a^b f(t)X(t)dt\right]=\int_a^b f(t)EX(t)dt$；

证明：$\because \int_a^b f(t)X(t)dt=\lim\limits_{\Delta\to0}\sum\limits_{k=1}^{n}f(u_k)X(u_k)(t_k-t_{k-1})$

$\therefore E\left[\int_a^b f(t)X(t)dt\right]$

$=E\lim\limits_{\Delta\to0}\sum\limits_{k=1}^{n}f(u_k)X(u_k)(t_k-t_{k-1})$

$=\lim\limits_{\Delta\to0}\sum\limits_{k=1}^{n}Ef(u_k)X(u_k)(t_k-t_{k-1})$

$=\int_a^b f(t)EX(t)dt$

7. 设 $\{X(t), a \leqslant t \leqslant b\}$ 是均方可导的随机过程，试证

$$\underset{t \to t_0}{\mathrm{l \cdot i \cdot m}}\, g(t)X(t) = g(t_0)X(t_0)$$

这里 $g(t)$ 是在区间 $[a,b]$ 上的连续函数.

证明：要证 $\underset{t \to t_0}{\mathrm{l.i.m}}\, g(t)X(t) = g(t_0)X(t_0)$

只需证 $\lim\limits_{t \to t_0} E|g(t)X(t) - g(t_0)X(t_0)|^2 = 0$

$g(t)X(t) - g(t_0)X(t_0) = g(t)X(t) - g(t)X(t_0) + g(t)X(t_0) - g(t_0)X(t_0)$

$= g(t)[X(t) - X(t_0)] + X(t_0)[g(t) - g(t_0)]$

$\therefore |g(t)X(t) - g(t_0)X(t_0)|^2 = g^2(t)[X(t) - X(t_0)]^2 + X^2(t_0)[g(t) - g(t_0)]^2$

$+ 2g(t)[X(t) - X(t_0)] \cdot [g(t) - g(t_0)] \cdot X(t_0)$

$\therefore E|g(t)X(t) - g(t_0)X(t_0)|^2 \leq g^2(t)E|X(t) - X(t_0)|^2 + [g(t) - g(t_0)]^2 EX^2(t_0)$

$+ 2|g(t)[g(t) - g(t_0)]| \sqrt{E|X(t) - X(t_0)|^2} \sqrt{E^2 X(t_0)}$

$\because \lim\limits_{t \to t_0} E|X(t) - X(t_0)|^2 = 0 \quad \lim\limits_{t \to t_0} g(t) = g(t_0)$

$\therefore \lim\limits_{t \to t_0} E|g(t)X(t) - g(t_0)X(t_0)|^2 = 0$ 即 $\underset{t \to t_0}{\mathrm{l.i.m}}\, g(t)X(t) = g(t_0)X(t_0)$

9. 设 $X(t) = S + Vt + At^2, t \geqslant 0$，其中 S, V, A 为相互独立的正态变量，试证 $X(t)$ 是一个正态过程.

证明： $S \sim N(u_1, \sigma_1^2) \quad V \sim N(u_2, \sigma_2^2) \quad A \sim N(u_3, \sigma_3^2)$

对于一维 $n=1$ 时 $\forall t \in (-\infty, +\infty)$ $E[X(t)] = E(S + Vt + At^2) = u_1 + u_2 t + u_3 t^2$

$D[X(t)] = D(S + Vt + At^2) = \sigma_1^2 + t^2\sigma_2^2 + t^4\sigma_3^2$

$\therefore X(t) \sim N(u_1 + u_2 t + u_3 t^2, \ \sigma_1^2 + t^2\sigma_2^2 + t^4\sigma_3^2)$

$n>1$ 时 对任意 n 个 t，取 $(t_1, t_2, \cdots, t_n)$

$E[X(t)] = (u_1 + u_2 t_1 + u_3 t_1^2, \ u_1 + u_2 t_2 + u_3 t_2^2, \cdots, u_1 + u_2 t_n + u_3 t_n^2)^T$

$$\begin{pmatrix} X(t_1) \\ X(t_2) \\ \vdots \\ X(t_n) \end{pmatrix} = \begin{pmatrix} 1 & t_1 & t_1^2 \\ 1 & t_2 & t_2^2 \\ \vdots & \vdots & \vdots \\ 1 & t_n & t_n^2 \end{pmatrix} \cdot \begin{pmatrix} S \\ V \\ A \end{pmatrix}$$

由对角线得

$$D[X(t)] = \begin{pmatrix} \sigma_1^2 + t_1^2\sigma_2^2 + t_1^4\sigma_3^2 \\ \sigma_1^2 + t_2^2\sigma_2^2 + t_2^4\sigma_3^2 \\ \vdots \\ \sigma_1^2 + t_n^2\sigma_2^2 + t_n^4\sigma_3^2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & t_1 & t_1^2 \\ 1 & t_2 & t_2^2 \\ \vdots & \vdots & \vdots \\ 1 & t_n & t_n^2 \end{pmatrix} \begin{pmatrix} \sigma_1^2 & & \\ & \sigma_2^2 & \\ & & \sigma_3^2 \end{pmatrix} \begin{pmatrix} 1 & 1 & \cdots & 1 \\ t_1 & t_2 & \cdots & t_n \\ t_1^2 & t_2^2 & \cdots & t_n^2 \end{pmatrix} = \begin{pmatrix} \sigma_1^2 + t_1^2\sigma_2^2 + t_1^4\sigma_3^2 & \cdots & \sigma_1^2 + t_1 t_n \sigma_2^2 + t_1^2 t_n^2 \sigma_3^2 \\ \vdots & \ddots & \\ \sigma_1^2 + t_1 t_n \sigma_2^2 + t_1^2 t_n^2 \sigma_3^2 & \cdots & \sigma_1^2 + t_n^2\sigma_2^2 + t_n^4\sigma_3^2 \end{pmatrix}$$

$\therefore X(t) \sim N\left((u_1 + u_2 t_1 + u_3 t_1^2, \cdots, u_1 + u_2 t_n + u_3 t_n^2)^T, (\sigma_1^2 + t_1^2\sigma_2^2 + t_1^4\sigma_3^2, \cdots, \sigma_1^2 + t_n^2\sigma_2^2 + t_n^4\sigma_3^2)^T\right)$

是正态过程

12. 证明：$f(t)X(t)$ 在区间 $[a,b]$ 上均方可积的充分条件是二重积分 $\int_a^b\int_a^b f(s)f(t)R_X(s,t)\mathrm{d}s\mathrm{d}t$ 存在，且有

$$E\left|\int_a^b f(t)X(t)\mathrm{d}t\right|^2=\int_a^b\int_a^b f(s)f(t)R_X(s,t)\mathrm{d}s\mathrm{d}t$$

证明：要证 $f(t)X(t)$ 在区间 $[a,b]$ 上均方可积

即证 $\lim\limits_{\Delta\to 0}\sum\limits_{k=1}^{n} f(u_k)X(u_k)(t_k-t_{k-1})$ 存在

只需证 $\lim\limits_{\Delta\to 0,\Delta'\to 0}\left(\sum\limits_{k=1}^{n} f(u_k)X(u_k)(t_k-t_{k-1})-\sum\limits_{l=1}^{m} f(v_l)X(v_l)(s_l-s_{l-1})\right)=0$

其中 $a=s_0<s_1<\cdots<s_m=b$ 为区间 $[a,b]$ 另一组分点，$s_{l-1}\le v_l\le s_l$

$\Delta'=\max\limits_{1\le l\le m}(s_l-s_{l-1})$

只需证 $\lim\limits_{\Delta\to 0,\Delta'\to 0} E\left|\sum\limits_{k=1}^{n} f(u_k)X(u_k)(t_k-t_{k-1})-\sum\limits_{l=1}^{m} f(v_l)X(v_l)(s_l-s_{l-1})\right|^2=0$

只需证 $\lim\limits_{\Delta\to 0,\Delta'\to 0}\Big[\sum\limits_{k=1}^{n}\sum\limits_{i=1}^{n} f(u_k)f(u_i)R_X(u_k,u_i)(t_k-t_{k-1})(t_i-t_{i-1})+$

$\sum\limits_{l=1}^{m}\sum\limits_{j=1}^{m} f(v_l)f(v_j)R_X(v_l,v_j)(s_l-s_{l-1})(s_j-s_{j-1})-$

$2\sum\limits_{k=1}^{n}\sum\limits_{l=1}^{m} f(u_k)f(v_l)R_X(u_k,v_l)(t_k-t_{k-1})(s_l-s_{l-1})\Big]=0$

由二重积分定义，上式左侧 $=\int_a^b\int_a^b f(s)f(t)R_X(s,t)\mathrm{d}s\mathrm{d}t$

$+\int_a^b\int_a^b f(s)f(t)R_X(s,t)\mathrm{d}s\mathrm{d}t$

$-2\int_a^b\int_a^b f(s)f(t)R_X(s,t)\mathrm{d}s\mathrm{d}t=0=$ 上式右侧

$\because \int_a^b\int_a^b f(s)f(t)R_X(s,t)\mathrm{d}s\mathrm{d}t$ 存在

$\therefore \int_a^b f(t)X(t)\mathrm{d}t$ 存在

$\therefore \lim\limits_{\Delta\to 0,\Delta'\to 0} E\left[\sum\limits_{k=1}^{n} f(u_k)X(u_k)(t_k-t_{k-1})\sum\limits_{l=1}^{m} f(v_l)X(v_l)(s_l-s_{l-1})\right]$ 存在

且等于 $E\left|\int_a^b f(t)X(t)\mathrm{d}t\right|^2$

$\therefore \int_a^b\int_a^b f(s)f(t)R_X(s,t)\mathrm{d}s\mathrm{d}t=E\left|\int_a^b f(t)X(t)\mathrm{d}t\right|^2$